

Optimierung und Entscheidungsunterstützung (OuE)

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Topics for today:

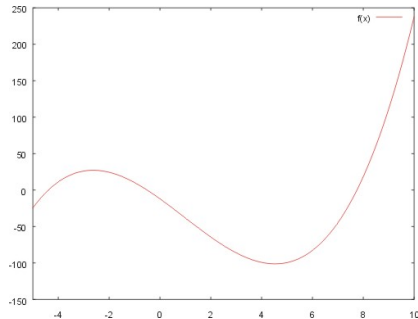
- 1 LP geometric interpretation
- 2 Towards the Simplex method for LP problems

Unconstrained optimization problems

Variables: x

Task: find values for the variables that minimize $f(x)$:

$$f(x) = a * x^3 + b * x^2 + c * x - 2 \quad (1)$$

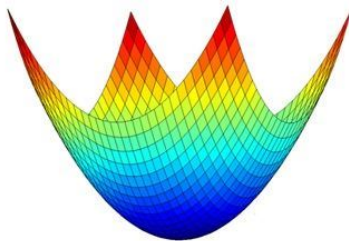


Unconstrained optimization problems

Variables: x, y

Task: find values for the variables that minimize $f(x, y)$:

$$f(x, y) = x^2 + y^2 \quad (2)$$



Unconstrained optimization problems

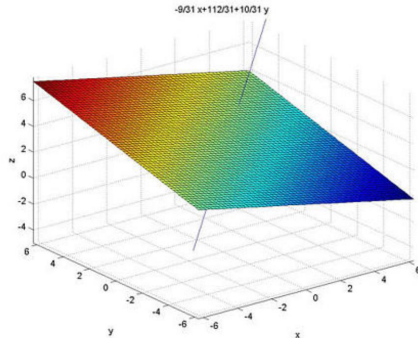
- no constraints on the variables
- function graph can have any shape
- solution methods: Gradient Descent, Newton's Method, Golden Section Search

Constraint optimization problems

- there are constraints on the variables
- there is a *feasible region* in the function graph
- special case today: linear optimization problems

Linear optimization problems

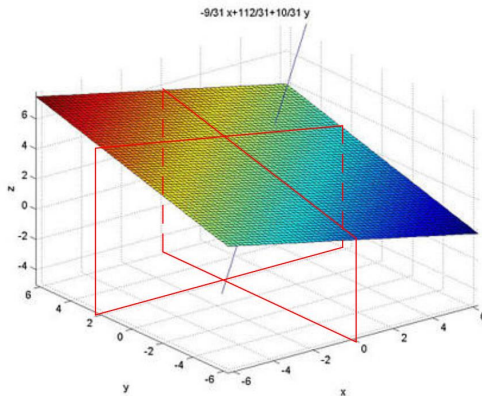
- feasible region is bounded by straight lines
- function graph is a line (1d), plane (2d) or hyperplane (3d)



Linear optimization problems

$$z = \frac{-9}{31}x + \frac{10}{31}y + \frac{112}{31} \quad (3)$$

s.t. $x > 0$, s.t. $y > 2$



Linear optimization problems

- In solving LP problems, the number of dimensions we need to consider is the number of variables.

Linear optimization problems

$$x \geq 0, y \geq 0$$

Maximize:

$$z = 15 * x_1 + 10 * y_1 \quad (4)$$

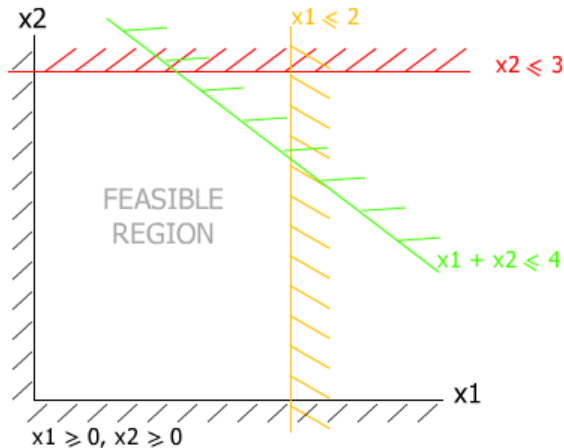
s.t.

$$x_1 \leq 2 \quad (5)$$

$$x_2 \leq 3 \quad (6)$$

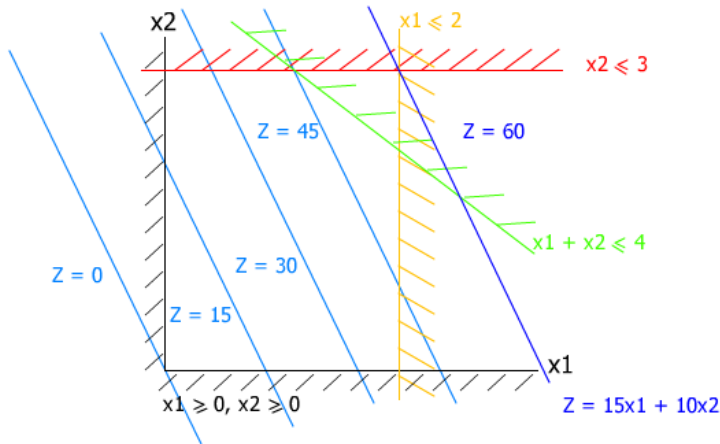
$$x_1 + x_2 \leq 4 \quad (7)$$

Linear optimization problems



This defines the feasible region.

Linear optimization problems

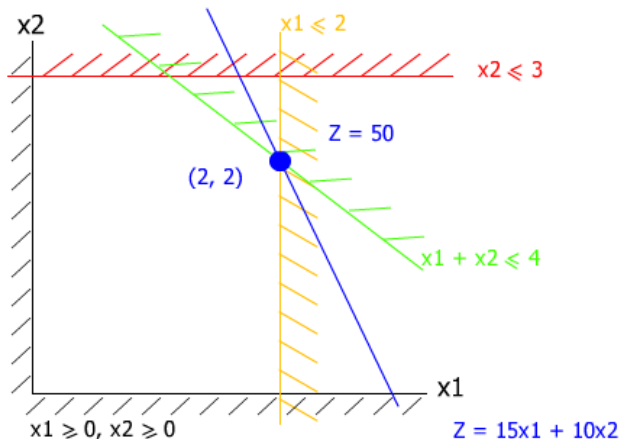


As we increase Z , we see the constant-profit line change.

Quelle: <http://optlab-server.sce.carleton.ca/>

POAnimations2007/Graph.html

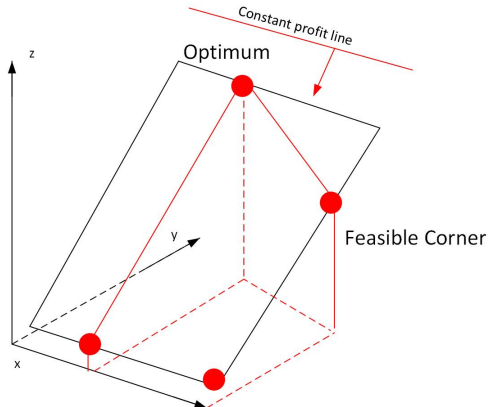
Linear optimization problems



Quelle: <http://optlab-server.sce.carleton.ca/P0Animations2007/Graph.html>

Linear optimization problems

A similar maximization task:



Linear optimization problems

Observation:

- Linear functions have a constant gradient

Observation 2:

- imagine the objective function as a line sinking from infinity (e.g. from the upper right to the lower left)
- where will it first bump into this feasible region defined by cornerpoints?
- → it will always first bump into the feasible region at a cornerpoint!

Where is the optimum?

A cornerpoint “sticks out farthest” in the direction of the sinking objective function line, hence first contact will be at a cornerpoint, **and this will define the optimum point.**

Key properties of linear programs

- ① **an** optimum solution to the LP is always at a cornerpoint.
- ② If a cornerpoint feasible solution has an objective function value that is **better than or equal** to all adjacent cornerpoint feasible solutions, then it is optimal.
- ③ There are a **finite number** of cornerpoint feasible solutions.

Key properties of linear programs

Conclusion

We need to look only at the **relatively small number of cornerpoints** of the feasible region instead of randomly sampling the infinite number of points on the interior of the feasible region.

Bird's-eye view of the Simplex Method

- 1. find any cornerpoint feasible solution.*
- 2. repeatedly move to a better adjacent cornerpoint feasible solution until no further better adjacent cornerpoint feasible solutions can be found.*

Finding Cornerpoints Algebraically

- real problems have hundreds, thousands, even millions of variables
- we need an algebraic way to find the cornerpoints.

Finding Cornerpoints Algebraically

First step:

- convert inequalities to equalities
- for this, we have to introduce *slack variables*

$x_1 \leq 2$ becomes $x_1 + s_1 = 2$, nonnegative slack variable s_1 is added.

Finding Cornerpoints Algebraically

$Z =$	original variables	slack variables	
	$15x_1 + 10x_2$		
	x_1	$+ s_1$	$= 2$
	x_2	$+ s_2$	$= 3$
	$x_1 + x_2$	$+ s_3$	$= 4$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

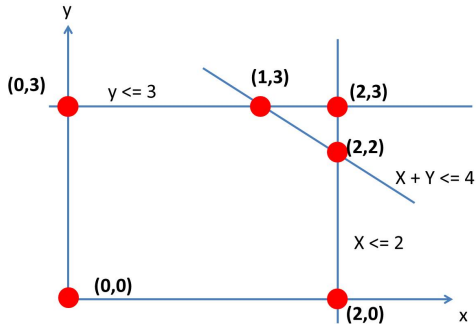
Figure 3.3: Equality version of the Acme Bicycle Company problem.

Finding Cornerpoints Algebraically

- now we have a system of linear equations
- we want to find the intersection of *some* of them
- for this, there are well-developed methods, such as Gaussian elimination.

Finding Cornerpoints Algebraically

6 intersection points



Question

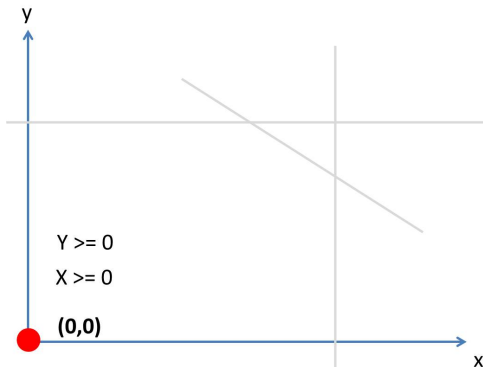
Which corner is infeasible?

Finding Cornerpoints Algebraically

- In each step, the solution of our equation system should be **one** of these intersection points.
- Obviously, we cannot produce all the intersection points at once.

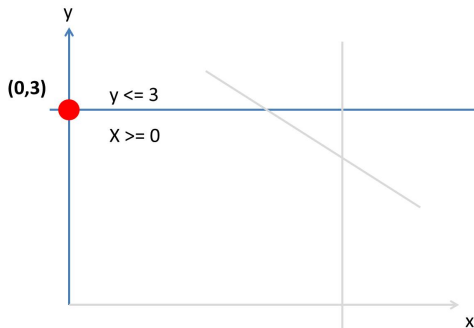
Finding Cornerpoints Algebraically

In each step, we *activate* two constraints:



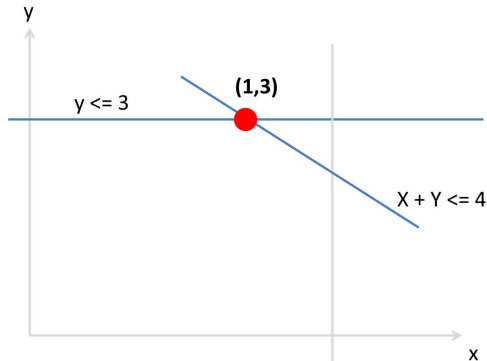
Finding Cornerpoints Algebraically

In each step, we *activate* two constraints:



Finding Cornerpoints Algebraically

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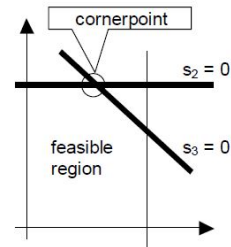
Finding Cornerpoints Algebraically

Observation

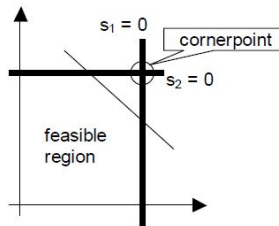
We only need the limiting values that fulfill a constraint.

- $x_1 = 0$ implies that you are on the limiting value of the $x_1 \geq 0$ constraint.
- $x_2 = 0$ implies that you are on the limiting value of the $x_2 \geq 0$ constraint.
- $s_1 = 0$ implies that you are on the line $x_1 = 2$.
- $s_2 = 0$ implies that you are on the line $x_2 = 3$.
- $s_3 = 0$ implies that you are on the line $x_1 + x_2 = 4$.

Finding Cornerpoints Algebraically



(a) feasible cornerpoint



(b) infeasible cornerpoint

Figure 3.4: Setting variables to zero defines a cornerpoint.

Finding Cornerpoints Algebraically

Observation

- $s_1 > 0$ implies that you are *outside* the line $x_1 = 2$.
- $s_2 > 0$ implies that you are *outside* the line $x_2 = 3$.
- $s_3 > 0$ implies that you are *outside* the line $x_1 + x_2 = 4$.

Finding Cornerpoints Algebraically

How do we *activate* and *deactivate* constraints?

Finding Cornerpoints Algebraically

Answer

Simplex

Simplex will set the values of two of the 5 variables to zero. This will make two of the constraints active, and it will thereby define a cornerpoint where these two constraints cross.

→ we solve the whole equation system to get the corner point.

Finding Cornerpoints Algebraically

Interpretation:

- For two constraints we force x_1 and x_2 to be on the line that is defined by the linear equation of these constraints $\rightarrow$ we get the intersection point we are interested in.
- The remaining constraints are fulfilled too, but x_1 and x_2 are outside the corresponding lines.
- However, their values fulfill the equations $\rightarrow$ ensured by the slack variables.

Finding Cornerpoints Algebraically

- Variables, that are set to zero are **nonbasic variables**

Remember

Nonbasic, variable set to zero, corresponding constraint is active.

Finding Cornerpoints Algebraically

Why two variables?

- we have 5 variables, but only 3 constraints
- this means that we can set the value of 2 of the variables arbitrarily (e.g. to 0)
- we say, the degree of freedom is 2

Finding Cornerpoints Algebraically

Example:

1. We set $s_2 = 0$, $s_3 = 0 \rightarrow$ nonbasic variables
2. We solve the equation system for x_1 , x_2 and $s_1 \rightarrow$ for the basic variables
3. We get $x_1 = 1$, $x_2 = 3$, $s_1 = 1$, $\rightarrow$ intersection point (1,3)!

	original variables	slack variables	
$Z =$	$15x_1 + 10x_2$		
	x_1	$+ s_1$	$= 2$
	x_2	$+ s_2$	$= 3$
	$x_1 + x_2$	$+ s_3$	$= 4$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

Figure 3.3: Equality version of the Acme Bicycle Company problem.

Moving to a Better Basic Feasible Solution

How to move to the next corner?

Observation

If you are at a feasible cornerpoint, then the easiest basic feasible solution to find next will be adjacent.

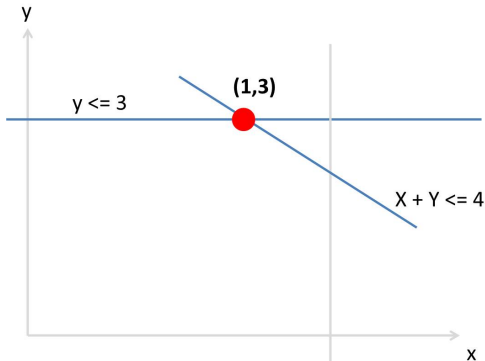
Moving to a Better Basic Feasible Solution

Red point

Nonbasic set = $\{s_2, s_3\}$, basic set = $\{x_1, x_2, s_1\}$

Adjacent point (invalid):

Nonbasic set = $\{s_1, s_2\}$, basic set = $\{x_1, x_2, s_3\}$



Moving to a Better Basic Feasible Solution

Moving to the next corner:

- One variable has to enter the basic set (*entering basic variable*).
- One variable has to leave the basic set (*leaving basic variable*).

Moving to a Better Basic Feasible Solution

However, just swapping a pair of variables between the basic and the nonbasic sets **is not sufficient**.

There are three conditions:

- the cornerpoints must be adjacent
- the cornerpoints must be feasible
- the new cornerpoint must have a better value of the objective function than the current cornerpoint.

Moving to a Better Basic Feasible Solution

Rules:

- ① Determine which nonbasic variable will increase the objective function value most swiftly if allowed to take on a positive value. This is called the entering basic variable because it is entering the basic set.
- ② One of the basic variables is forced to a value of zero. This is called the leaving basic because it is leaving the basic set.

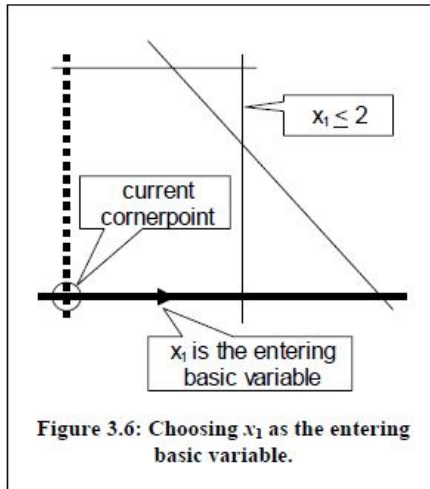
Moving to a Better Basic Feasible Solution

Assuming we are at the origin (0,0):

$Z = 15 * x_1 + 10 * x_2$, so it is immediately obvious that x_1 provides the swiftest rate of increase in Z .

- **Entering basic variable:** x_1
- **leaving basic variable:** $s_1 \rightarrow$ activates constraint $x_1 \leq 2$

Linear optimization problems



Das war's für heute :-)